

Amalgamated Products of Groups II: Measures of Random Normal Forms

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Abstract

Let $G = A *_C B$ be an amalgamated product of finite rank free groups A , B and C . We introduce atomic measures and corresponding asymptotic densities on a set of normal forms of elements in G . We also define two strata of normal forms: the first one consists of regular (or stable) normal forms, and second stratum is formed by singular (or unstable) normal forms. In a series of previous work about classical algorithmic problems, it was shown that standard algorithms work fast on elements of the first stratum and nothing is known about their work on the second stratum. In theorems A and B of this paper we give probabilistic and asymptotic estimates of these strata.

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Introduction. Let F be a free group with basis X , $|X| < \infty$. In [3, 7] the authors introduced a technique which help to analyze complexity of algorithmic problems for finitely generated groups of type $G = F/N$. In practical computations elements of G are usually written in a form of freely-reduced words in X (normal forms in G), and therefore all computations in G take place in F . To analyze a given algorithmic problem in G one should have:

- (i) satisfactory normal forms for elements of G ;
- (ii) convenient generators of random elements of G in normal forms;
- (iii) atomic and probability measures on F for measuring elements and subsets of F ;
- (iv) results on stratification of inputs of algorithms (i.e. normal forms) at least on two strata: stratum of regular (or stable) elements on which

algorithms work fast, for example, in polynomial time, and another one of singular and unstable elements on which the result of the algorithms work is unknown or it works slow;

(v) asymptotic and probabilistic tools of estimation of these strata.

Here we lay out briefly, what has been done in previous papers of the authors (and their coauthors) and main results of this paper.

First of all, we work with groups representable in a form of some free construction, mostly as a free product with amalgamation, i.e. $G = A *_C B$. It guarantees the existence of convenient normal forms of elements in G if these forms exist in A and B . If A, B, C are free groups of finite ranks, we construct (see section 3 for details) four generators of random elements in different normal forms and specify probabilities to obtain such elements.

In [3] there was constructed a series of atomic measures $\{\mu_s | 0 < s < 1, s \in \mathbb{R}^+\}$ on F with the help of a no-return random walk on the Cayley graph of $F = F(X)$. It allows us, firstly, to make an asymptotic classification of subsets of F , and, secondly, to prove an important result about asymptotic properties of regular subsets of F (i.e. sets accepted by finite automaton).

Theorem 1. [3, Theorem 3.2]. *Let R be a regular subset of F . Then R is thick if and only if its prefix closure \overline{R} contains a cone¹.*

In [7] this result was generalized to a stronger form:

Theorem 2. [7, Theorem 5.4]. *Let R be a regular subset of a prefix-closed regular set L in a finite rank free group F . Then either the prefix closure \overline{R} of R in L contains a non-small L -cone or \overline{R} is exponentially λ_L -measurable.*

This theorem plays a significant role in the proof of the main results (Theorems A and B) of this paper.

Stratification of inputs was described in the papers [4, 7]. In particular, we estimate the sizes of strata in Schreier systems of representatives (transversals) in a free group. The main result here is a following

Theorem 3. [7, Theorem 5.4]. *Let C be a finitely generated subgroup of infinite index in $F(X)$ and S be a Schreier transversal for C . Then sets of all singular representatives S_{sin} and unstable representatives S_{uns} are exponentially negligible relative to S .*

An example of a stratification of inputs is an algorithm deciding the Conjugacy Search problem for elements of a group given in [4]:

¹All necessary definitions we give below in Section 4.

Theorem 4. [4, Corollary 4.19]. *Let $G = A *_C B$ be a free product of finitely generated free groups A and B with amalgamated finitely generated subgroup C . Then the Conjugacy Search Problem in G is decidable for all cyclically reduced canonical forms G .*

Here the cyclically reduced regular elements form the first stratum and complementary set form the second one.

Main results of this paper is related to item (v) of the research program described above and is contained in the two theorems:

Theorem A. *Let $G = A *_C B$ be an amalgamated product, where A, B, C are free groups of finite rank. Then for every set of normal forms $\mathcal{NF} = \{\mathcal{EF}, \mathcal{RF}, \mathcal{CNF}, \mathcal{CRF}\}$*

- (i) *If C has a finite index in A and in B , then every normal form is singular and unstable, i.e. $\mathcal{NF}_{\text{sin}} = \mathcal{NF}_{\text{uns}} = \mathcal{NF}$;*
- (ii) *If C of infinite index either in A or in B , then \mathcal{NF}_r and \mathcal{NF}_s are exponentially μ -generic relative to \mathcal{NF} , and $\mathcal{NF}_{\text{sin}}$ and $\mathcal{NF}_{\text{uns}}$ are exponentially μ -negligible relative to \mathcal{NF} in the following cases:*
 - (ii.1) *μ is defined by pseudo-measures μ_A and μ_B , which are cardinality functions on A and B correspondingly; in this case ρ_μ is a bidimensional asymptotic density;*
 - (ii.2) *μ is defined by atomic probability measures $\mu_{A,l}$ and $\mu_{B,l}$ on A and B correspondingly; in this case ρ^C is a bidimensional Cesaro asymptotic density.*

Theorem B. *Let $G = A *_C B$ be an amalgamated product, where A, B, C are free groups of finite rank. If C of infinite index either in A or in B , then sets of all unstable $\mathcal{NF}_{\text{uns}}$ and all singular $\mathcal{NF}_{\text{sin}}$ normal forms are exponentially $\lambda_{\mathcal{NF}}$ -measurable, where $\mathcal{NF} = \{\mathcal{EF}, \mathcal{RF}, \mathcal{CNF}, \mathcal{CRF}\}$.*

Specifically, the current papers relation to the other work of this series is the following. Here, we work with a group $G = A *_C B$ where $A = F(X)$, $B = F(Y)$, and C are free groups of finite ranks and given atomic measures μ_A and μ_B on A and B correspondingly, as well as asymptotic densities induced by these measures. It was necessary to define correctly bidimensional measure μ on $F = A * B$ and asymptotic densities of subsets of $A * B$. We do it in Section 2 of this paper.

We follow [11, 12] on the subject of group theory; [6] for formal languages; [10, 14] for random walks. We essentially use the terminology of the papers [3, 4, 7].

1 Preliminaries

In this section we recap some of the definitions and facts about free products with amalgamation. We refer to [12] for more details. Let A, B, C be groups and $\varphi : C \rightarrow A$ and $\psi : C \rightarrow B$ be monomorphisms. Then one can define a group $G = A *_C B$, called the amalgamated product of A and B over C (the monomorphisms φ, ψ are usually suppressed from notation). If A and B are given by presentations $A = \langle X | R_A = 1 \rangle$, $B = \langle Y | R_B = 1 \rangle$, and a generating set Z is given for the group C , then the group G has a presentation

$$G = \langle X \cup Y | R_A = 1, R_B = 1, \varphi(z) = \psi(z), z \in Z \rangle. \quad (1)$$

If we denote $\varphi(z) = u_z(x)$, $\psi(z) = v_z(y)$ then G has a presentation

$$G = \langle X \cup Y | R_A = 1, R_B = 1, u_z(x) = v_z(y), (z \in Z) \rangle.$$

Groups A and B are called factors of the amalgamated product $G = A *_C B$; they are isomorphic to subgroups in G generated respectively by X and Y . We will identify A and B with these subgroups via evident maps.

Denote by S and T the fixed systems of right coset representatives of C in A and B respectively. Throughout this paper we assume that the representative of C is the identity element 1. For an element $g \in (A \cup B) \setminus C$ we define $F(g) = A$ if $g \in A$ and $F(g) = B$ if $g \in B$.

Here are the four main methods to represent an element of G :

1. by a word in the alphabet $X \cup X^{-1} \cup Y \cup Y^{-1}$:

According to the presentation (1) of G , every nontrivial element $g \in G$ can be written in the form

$$g = g_1 g_2 \dots g_n, \quad (2)$$

where g_1, \dots, g_n are reduced words in $X \cup X^{-1}$ or in $Y \cup Y^{-1}$, and if $F(g_i) = A$ then $F(g_{i+1}) = B$, and vice versa. Obviously, the form (2) of an element g is not unique; moreover, the number n of multipliers

corresponding to different representations of g in the form (2) can vary ad libitum.

2. in the unique canonical normal form (see [12] for details):

$$g = cp_1p_2 \dots p_l, \quad (3)$$

where $c \in C$, $p_i \in (S \cup T) \setminus \{1\}$, and $F(p_i) \neq F(p_{i+1})$, $i = 1, \dots, l$, $l \geq 0$.

If the **Coset Representative Search Problem** (see, for example, [4] for details about the algorithmic problems in groups) is decidable for C in A and B then (3) can be computed from (2) effectively.

3. in the reduced form:

$$g = cg_1g_2 \dots g_k \quad (4)$$

where $c \in C$, $g_i \in (A \cup B) \setminus C$ and $F(g_i) \neq F(g_{i+1})$, $i = 1, \dots, k$, if $k \geq 0$. This form may not be unique, but the number k is uniquely determined by g . For technical reasons, we will use a slightly different definition of a reduced form as well. Namely, the element $g \in G$ is written in a reduced form, if

$$g = g_1g_2 \dots g_k \quad (5)$$

where $g_i \in (A \cup B) \setminus C$ and $F(g_i) \neq F(g_{i+1})$, $i = 1, \dots, k$, if $k \geq 1$ and $g = c$, if $k = 0$.

Obviously, both definitions (4) and (5) are equivalent. Moreover, if the **Membership Problem** for C in A and B is decidable, then (5) can be computed from (2) effectively. Every element in the reduced form (4) is a conjugate of an element

4. in the cyclically reduced form:

$$g = cg_1 \dots g_k \quad (6)$$

The form (6) is called *cyclically reduced form* of element g , if

- (i) $k = 0$, i.e. $g = c \in C$;

- (ii) $k = 1$, then every $g \in A \cup B$ that is not a conjugate of an element in C ;
- (iii) $k > 1$, then every g , such as k is even.

We will refer to an element $g \in G$ as an element in normal form throughout the paper if it has one of the forms (2), (3), (5) or (6) and it doesn't matter which one is chosen. Though, we mention that the numbers n, k, l of the factors in each representation of an element in different normal forms are different in general (and some time are not even unique), we will refer further to such a number as to length of the representation of g in normal form and denote it by $s(g)$.

1.1 Measuring and comparing subsets of free group

In this section we recap some crucial facts about measures in free group of finite rank $F(X)$; in more details you can find this information in [3, 7].

Let $\mathcal{P}(F)$ be the set of all subsets of $F = F(X)$ and $\mathcal{A} \subset \mathcal{P}(F)$. A real-valued non-negative additive function $\mu : \mathcal{A} \rightarrow \mathbb{R}^+$ is called a *pseudo-measure* on F . If \mathcal{A} is a subalgebra of $\mathcal{P}(F)$, then μ is a *measure*.

Let $F = F(X)$ be a free group. Denote by S_n and B_n correspondingly the sphere and the ball of radius n in F . Let μ be an atomic pseudo-measure on F . Recall, that a measure μ on the countable set P called is *atomic* if every subset $Q \subseteq P$ is measurable; it also holds when $\mu(Q) = \sum_{q \in Q} \mu(q)$.

For a set $R \subseteq F$ we define its *spherical asymptotic density* relative to μ as the following limit:

$$s\rho_\mu(R) = \overline{\lim}_{n \rightarrow \infty} s\rho_n(R),$$

$$\text{where } s\rho_n(R) = \frac{\mu(R \cap S_n)}{\mu(S_n)}.$$

Similarly, one can define the *ball asymptotic density* of R relative to μ :

$$b\rho_\mu(R) = \overline{\lim}_{n \rightarrow \infty} b\rho_n(R),$$

where $b\rho_n(R) = \frac{\mu(R \cap B_n)}{\mu(B_n)}$. We formulate below one very useful fact about the connection between spherical and ball asymptotic densities, a proof can be found, for example, in [8, Lemma 3.2].

Lemma 1.1. *Let μ be a pseudo-measure on F . Suppose that $\lim_{n \rightarrow \infty} \mu(B_n) = \infty$.*

Then for any subset $R \subseteq F$ if the spherical asymptotic density $s\rho_\mu(R)$ exists, then the ball asymptotic density $b\rho_\mu(R)$ also exists and

$$b\rho_\mu(R) = s\rho_\mu(R).$$

Further, let μ be a pseudo-measure on F and $\rho_\mu(R)$ be a spherical or ball asymptotic density.

We say that a subset $R \subseteq F$ is *generic* relative to μ , if the limit $\lim_{n \rightarrow \infty} s\rho_n(R)$ exists and $\rho_\mu(R) = 1$, and *negligible* relative to μ , if $\rho_\mu(R) = 0$.

Further, we say R is *exponentially generic* relative to μ if there exists a positive constant $\delta < 1$ such that $1 - \delta^n < s\rho_n(R) < 1$ for sufficiently large n . Meanwhile, if $s\rho_n(R) < \delta^n$ for all large enough n , then R is *exponentially negligible* relative to μ .

For example, if μ is the cardinality function, i.e. $\mu(A) = |A|$, then we obtain standard asymptotic density functions on F . We will use the notation $\rho(R)$ throughout the paper to denote the standard spherical asymptotic density of R in F relative to the cardinality function; we will also omit the "cardinality function" whenever possible. It is not also hard to extend such a definition for an asymptotic density of set R relative to set R_1 (see [7] and Section 4.4 for details). We will use the notation $\rho_\mu(R, R_1)$ for this asymptotic density.

Further, we will be interested on a special kind of measure in F , studied in details in a lot of papers (see, for example, [1, 3, 7]). Namely, consider a so-called *frequency measure* on $R \subseteq F = F(X)$:

$$\lambda(R) = \sum_{n=0}^{\infty} f_n(R), \text{ where } f_n(R) = \frac{|R \cap S_n|}{|S_n|},$$

and $f_n(R)$ are called *frequencies* of elements from R among the words of (freely-reduced) length n in F . This measure is not probabilistic, since, for instance, $\lambda(F) = \infty$, moreover, λ is additive, but not σ -additive.

Also, frequencies of R define a well-studied asymptotic density called *Cesaro asymptotic density*. Namely, it is the *Cesaro limit* of frequencies for R :

$$\rho^c(R) = \lim_{n \rightarrow \infty} \frac{1}{n} (f_1(R) + \cdots + f_n(R)). \quad (7)$$

Sometimes it is more sensitive than the standard asymptotic density ρ (see, for example, [3, 14]). However, if $\lim_{n \rightarrow \infty} f_n(R)$ exists (hence is equal to $\rho(R)$), then $\rho^c(R)$ also exists and $\rho^c(R) = \rho(R)$.

1.2 Stratification and measuring of Schreier systems of representatives in free group

In this section we give some information about Schreier transversals (see [7] for details) in free groups and also the definitions of regular and stable normal forms of elements in free product with amalgamation G .

Following [9], we associate with C two graphs: the *subgroup graph* $\Gamma = \Gamma_C$ and the Schreier graph $\Gamma^* = \Gamma_C^*$. Recall that Γ is a finite connected digraph with edges labeled by elements from X and a distinguished vertex (based-point) 1_C , satisfying the following two conditions. Firstly, Γ is folded, i.e., there are no two edges in Γ with the same label and having the same initial or terminal vertices. Secondly, Γ accepts precisely the reduced words in $X \cup X^{-1}$ that belong to C .

The *Schreier graph* $\Gamma^* = \Gamma_C^*$ of C is a connected labeled digraph with the set $\{Cu \mid u \in F\}$ of right cosets of C in F as the vertex set, and such that there is an edge from Cu to Cv with a label $x \in X$ if and only if $Cux = Cv$. One can describe the Schreier graph Γ^* as obtained from Γ by the following procedure. Let $v \in \Gamma$ and $x \in X$ such that there is no outgoing or incoming edge at v labeled by x . For every such vertex v and $x \in X$ we attach to v a new edge e (correspondingly, either outgoing or incoming) labeled x with a new terminal vertex u (not in Γ). Then we attach to u the Cayley graph $C(F, X)$ of F relative to X (identifying u with the root vertex of $C(F, X)$), and then we fold the edge e with the corresponding edge in $C(F, X)$ (that is labeled x and is incoming to u). Observe, that for every vertex $v \in \Gamma^*$ and every reduced word w in $X \cup X^{-1}$ there is a unique path Γ^* that starts at v and has the label w . By p_w we denote such a path that starts at 1_C , and by v_w the end vertex of p_w .

Consider a set of right representatives of C in $F = F(X)$; we will call it the *transversal* of C . Recall, that a transversal S of C is termed *Schreier* if every initial segment of a representative from S belongs to S . In [7] was shown, that there is one-to-one correspondence between the set of every Schreier transversal S of C and the set of all spanning subtrees Γ^* . In particular, it means that we can treat with every representative $s \in S$ as with label of a

path in some (fixed) spanning subtree of Γ^* .

Also, we have a classification of representatives of C in $F = F(X)$ from [7]:

Definition 1.2. Let S be a transversal of C .

- A representative $s \in S$ is called *internal* if the path p_s ends in Γ , i.e., $v_s \in V(\Gamma)$. By S_{int} we denote the set of all internal representatives in S . Elements from $S_{\text{ext}} = S \setminus S_{\text{int}}$ are called *external* representatives in S .
- A representative $s \in S$ is called *singular* if it belongs to the generalized normalizer of C :

$$N_F^*(C) = \{f \in F \mid f^{-1}Cf \cap C \neq 1\}.$$

All other representatives from S are called *regular*. By S_{sin} and, respectively, S_{reg} we denote the sets of singular and regular representatives from S .

- A representative $s \in S$ is called *stable* if $sc \in S$ for any $c \in C$. By S_{st} we denote the set of all stable representatives in S , and $S_{\text{uns}} = S \setminus S_{\text{st}}$ is the set of all *unstable* representatives from S .

Frontier vertex v_u of $V(\Gamma)$ is a vertex $v_u \in V(\Gamma^*) \setminus V(\Gamma)$ such that v_u incident to an edge e of Γ^* , which initial or terminal vertex already in $V(\Gamma)$. A *cone* $C(u)$ is a subset of F of type $\{w \in F : w = uf \text{ and } uv \text{ is a reduced word}\}$.

The following proposition about the structure of all singular and unstable representatives was shown in [7]:

Proposition 1.3. [7, Proposition 3.5]. Let S be a Schreier transversal for C , C has an infinite index in F and $S = S_{T^*}$ for some spanning subtree T^* of Γ^* . Then the following hold:

- 1) $|S_{\text{int}}| = |V(\Gamma)|$.
- 2) S_{ext} is the union of finitely many conis $C(u)$, where v_u are frontier vertices of Γ .
- 3) S_{sin} is contained in a finite union of double cosets $Cs_1s_2^{-1}C$ of C , where $s_1, s_2 \in S_{\text{int}}$.

- 4) S_{uns} is a finite union of left cosets of C of the type $s_1 s_2^{-1} C$, where $s_1, s_2 \in S_{\text{int}}$.
- 5) (see [Proposition 3.9, [7]]) $S_{\text{sin}} \subseteq S_{\text{uns}}$.

In [7] it was also shown that the sets of all singular and unstable representatives forms an exponentially negligible part of the Schreier transversals:

Corollary 1.4. [7, Corollary 5.12]. Let C be a finitely generated subgroup of infinite index in $F(X)$ and S a Schreier transversal for C . Then sets of singular representatives S_{sin} and unstable representatives S_{uns} are exponentially negligible in S .

Following the idea to split the set of all normal forms of elements in $G = A *_C B$ into "bad" and "good" components, we introduce the following definitions.

We say that an element $g \in F(X)$ is regular (stable and so on), if it can be decomposed into a form $g = cs$, $c \in C$, $s \in S$ and s is regular, stable etc.

Definition 1.5. An element $g \in G$ in normal form (2), (3), (5) or (6) is called *regular* if at least one of elements g_i or p_i ; $i = 1, \dots, s(g)$ is regular. Otherwise g is called *singular*.

Definition 1.6. An element $g \in G$ in normal form (2), (3), (5) or (6) is called *stable* if at least one of elements g_i or p_i ; $i = 1, \dots, s(g)$ is stable. Otherwise g is called *unstable*.

In main the Theorems A and B of this paper we estimate sizes of stable, unstable, regular and singular components in the set of all normal forms, and these notions will be very important for the rest of the paper.

2 Asymptotic densities on free products of subsets

In this Section our goal is a definition of asymptotic densities on subsets of $G = A *_C B$, induced by different types of measures on factors A and B , introduced in Section 1.1. In turn, it constrain us to define measures on free product of subsets of $F = A * B$.

2.1 Free product of subsets

Let $F = A * B$ be a free product of finitely generated groups A and B , and A_0 is a nonempty subset of A , B_0 is a nonempty subset of B . Then a *free product* $A_0 * B_0$ is a set of all elements in F having a form $f = f_1 f_2 \dots f_k$, where $n \geq 1$; $f_i \in A_0 \cup B_0$, $i = 1, \dots, k$, $f_i \neq 1$ if $i \geq 2$ and for all $i = 1, \dots, k - 1$ elements $F(f_i) \neq F(f_{i+1})$. Every nontrivial element $f \in F$ can be written in a freely-reduced form

$$f = f_1 f_2 \dots f_k, \quad (8)$$

where f_1, \dots, f_n are reduced words in $X \cup X^{-1}$ or in $Y \cup Y^{-1}$, and if $F(f_i) = A$, then $F(f_{i+1}) = B$ and vice versa. For every such nontrivial $f \in F$ set $s(f) = k$; let $s(1) = 0$. We will denote by $|f|$, $|f_i|$ the number of letters in alphabet $X \cup X^{-1}$ or $X \cup X^{-1} \cup Y \cup Y^{-1}$ in a freely-reduced form of words f, f_i .

Let μ_A and μ_B be atomic pseudo-measures on A and B correspondingly and $\mu_A(1) = \mu_B(1)$. We also fix a probability distribution $\theta : \mathbb{N} \rightarrow \mathbb{R}^+$, in particular, $\sum_{k=1}^{\infty} \theta(k) = 1$. We define an atomic measure μ on F in the following manner:

$$\mu(f) = \frac{1}{2} \theta(k) \mu_{F_1}(f_1) \dots \mu_{F_k}(f_k), \quad (9)$$

where f is written in a freely-reduced form (8).

For a subset $R \subseteq F$ set

$$\mu(R) = \sum_{f \in R} \mu(f).$$

We will say, that R is a μ -measurable set, if $\mu(R) < \infty$. Denote by M_μ the set of all μ -measurable subsets of F :

$$M_\varphi = \{R \subseteq F | \mu(R) < \infty\}.$$

Lemma 2.1. *Let $F = A * B$ be a free product of finitely generated groups A and B .*

- 1) *If μ_A, μ_B are atomic pseudo-measures on A and B correspondingly, then measure μ on F defined above is an atomic pseudo-measure on F and $M_{\mu_A} \subset M_\mu, M_{\mu_B} \subset M_\mu$.*
- 2) *If μ_A, μ_B are atomic probability measures on A and B correspondingly, then μ is an atomic probability measure on F .*

Proof. Claim 1) is straightforward. Let us prove 2). Split F into layers:
 $F = F_1 \sqcup F_2 \sqcup F_3 \dots$, where

$$F_i = \{f \in F \mid f \text{ in a freely-reduced form (8) and } s(f) = i\}.$$

$$\begin{aligned} \text{Then } \mu(F) &= \sum_{i=1}^{\infty} \mu(F_i) = \\ &= \sum_{i=1}^{\infty} \frac{1}{2} \theta(i) \left(\mu_A^{[\frac{i+1}{2}]}(A) \mu_B^{i-[\frac{i+1}{2}]}(B) + \mu_A^{i-[\frac{i+1}{2}]}(A) \mu_B^{[\frac{i+1}{2}]}(B) \right) = \\ &= \sum_{i=1}^{\infty} \theta(i) = 1. \end{aligned}$$

•

Example. Suppose μ_A, μ_B are pseudo-measures on A and B , defined by cardinality functions on A and B , and $\theta(k) = \frac{6}{\pi^2 k^2}$ is a probability distribution on \mathbb{N} . Then $M_{\mu_A} = \mathcal{F}(A)$ and $M_{\mu_B} = \mathcal{F}(B)$, where $\mathcal{F}(A)$ and $\mathcal{F}(B)$ are sets of all finite subsets of A and B . However, $M_\mu \supset \mathcal{F}(F)$ is a strict inclusion. Indeed, let $R \subseteq F$ and $R_k = R \cap F_k$. Then $R \in M_\mu$ iff row $\sum \frac{|R_k|}{k^2}$ converges.

We shall describe below several methods to define asymptotic density of subsets in $F = A * B$.

2.2 Bidimensional asymptotic density

Let $T = A_0 * B_0 \subseteq F = A * B$. For a pair of natural numbers (n, k) we define (n, k) -ball:

$$T_{n,k} = \{f = f_1 \dots f_k \in T : s(f) = k, |f_i| \leq n, i = 1, \dots, k\}.$$

We call $T = \bigcup_{k=0, n=0}^{\infty} T_{n,k}$ the *bidimensional decomposition* of T . Bidimensional decompositions help us to analyze asymptotic behavior of subsets of T and other subsets of F relative to T . For a set $Q = A_1 * B_1$ in F a function $(n, k) \rightarrow \mu(Q \cap T_{n,k})$ is called the growth function of Q in T , and a function $(n, k) \rightarrow \rho_\mu^{n,k}(Q, T) = \frac{\mu(Q \cap T_{n,k})}{\mu(T_{n,k})}$ is called the frequency function of Q relative to T .

By *direction function* $d(n, k)$ we mean one-to-one correspondence between n and k which parametrize a path from $(1, 1)$ to (∞, ∞) such that arguments

n and k tends to ∞ while $d(n, k) \rightarrow \infty$. Let $d(n, k)$ be some direction function. Asymptotic behavior of Q relative to T we will characterize by a *bidimensional asymptotic density*, which determines as following limit:

$$\rho_\mu(Q, T) = \overline{\lim}_{d(n, k) \rightarrow \infty} \rho_\mu^{n, k}(Q, T).$$

If this limit exists and does not depend on a choice of a direction function, we denote it by $\rho_\mu^e(Q, T)$. We say that Q is μ -generic relative to T , if $\rho_\mu^e(Q, T) = 1$, and μ -negligible relative to T , if $\rho_\mu^e(Q, T) = 0$.

Further, we say that Q is *exponentially μ_n -generic* relative to T if Q is μ -generic and there exists a positive constant $\delta < 1$ such that $1 - \delta^{nk} < \rho_\mu^{n, k}(Q, T) < 1$ for all large enough n and k . Meanwhile, if Q is μ -negligible and $\rho_\mu^{n, k}(Q, T) < \delta^{nk}$ for sufficiently large n, k then Q is *exponentially μ_n -negligible* relative to T .

We will use below other notions of exponentially generic and negligible sets relative to measure μ , that will allow us to obtain in Section 4 more rough, but at the same time, more general estimates on subsets of normal forms. Namely, Q is said to be *exponentially μ -generic* relative to T if Q is μ -generic and there exists a positive constant $\delta < 1$ such that $1 - \delta^k < \rho_\mu^{n, k}(Q, T) < 1$ for large enough k . If Q is μ -negligible and $\rho_\mu^{n, k}(Q, T) < \delta^k$ for sufficiently large k , then Q is *exponentially μ -negligible* relative to T .

Denote $(T)_n = \{f = f_1 \dots f_k \in T \mid |f_i| = n, i = 1, \dots, k\}$, $(T)_{\leq n} = \{f = f_1 \dots f_k \in T \mid |f_i| \leq n, i = 1, \dots, k\}$, and $(T)^k = \{f \in T \mid s(f) = k\}$. We shall use the same notation for subsets of groups A and B when there is no ambiguity, i.e. notation $(A_i)_n = (A)_n \cap A_i$ (or $(B_i)_n = (B)_n \cap B_i$) for the sphere of radius n in a subsets of A (or B) and $(A_i)_{\leq n} = (A)_{\leq n} \cap A_i$ (or $(B_i)_{\leq n} = (B)_{\leq n} \cap B_i$) for the ball of radius n in corresponding sets.

The following proposition will be useful in the sequel.

Proposition 2.2. Let $F = A * B$ be a free product of free groups A and B of finite ranks, and $A_1 \subseteq A_0 \subseteq A$, $B_1 \subseteq B_0 \subseteq B$, and let $T = A_0 * B_0$, and $Q = A_1 * B_1$.

1. Suppose μ_A and μ_B are atomic probability measures on A and B correspondingly and μ is a pseudo-measure on F , defined above and let $\rho_{\mu_A}(A_1, A_0)$ and $\rho_{\mu_B}(B_1, B_0)$ exist. If there is a constant $\delta, 0 < \delta < 1$ such that for all $n > n_0$ either $\frac{\mu_A((A_1)_{\leq n})}{\mu_A((A_0)_{\leq n})} < \delta^n$ or $\frac{\mu_B((B_1)_{\leq n})}{\mu_B((B_0)_{\leq n})} < \delta^n$, then Q is exponentially μ_n -negligible relative to T .

2. Suppose μ_A and μ_B are atomic probability measures on A and B correspondingly and μ is a pseudo-measure on F , defined above. Let $\rho_{\mu_A}(A_1, A_0)$ and $\rho_{\mu_B}(B_1, B_0)$ exist and at least one of them less than 1. Then Q is exponentially μ -negligible relative to T .
3. Suppose μ_A and μ_B are pseudo-measures on A and B defined by cardinality functions and μ is a pseudo-measure on F , defined above. Let $\rho(A_1, A_0)$ and $\rho(B_1, B_0)$ exist and at least one of them less than 1. Then Q is exponentially μ -negligible relative to T .

Proof. To prove first claim, suppose $\frac{\mu_A((A_1)_{\leq n})}{\mu_A((A_0)_{\leq n})} < \delta^n$; then by definition for all $n \geq n_0$ we obtain $\frac{\mu(Q \cap T_{n,k})}{\mu(T_{n,k})} =$

$$= \begin{cases} k = 2t & \frac{(\mu_A((A_1)_{\leq n}))^t (\mu_B((B_1)_{\leq n}))^t}{(\mu_A((A_0)_{\leq n}))^t (\mu_B((B_0)_{\leq n}))^t} \\ k = 2t + 1 & \frac{(\mu_A((A_1)_{\leq n}))^t (\mu_B^t((B_1)_{\leq n}))^t (\mu_A((A_1)_{\leq n}) + \mu_B((B_1)_{\leq n}))}{(\mu_A((A_0)_{\leq n}))^t (\mu_B((B_0)_{\leq n}))^t (\mu_A((A_0)_{\leq n}) + \mu_B((B_0)_{\leq n}))} \end{cases} \quad (10)$$

Since $\rho_{\mu_B}(B_1, B_0)$ exists there is a natural number n_1 such that

$$\frac{\mu(Q \cap T_{n,k})}{\mu(T_{n,k})} \leq \delta^{nt} < \delta^{n(t-1)} < \delta^{n(k/2-2)}$$

for all $n \geq n_1$. Therefore, the limit $\lim_{d(n,k) \rightarrow \infty} \frac{\mu(Q \cap T_{n,k})}{\mu(T_{n,k})}$ exists and does not depend on a $d(n, k)$. It equals to zero and moreover, Q is exponentially μ_n -negligible relative to T with a $\delta' = \delta^{1/2}$.

To prove 2), suppose that $\rho_{\mu_A}(A_1, A_0) < 1$. By simple observation since the limit exists

$$\rho_{\mu_A}(A_1, A_0) = \lim_{n \rightarrow \infty} \frac{\mu_A((A_1)_n)}{\mu_A((A_0)_n)} = \lim_{n \rightarrow \infty} \frac{\mu_A((A_1)_{\leq n})}{\mu_A((A_0)_{\leq n})}$$

and therefore

$$\frac{\mu_A((A_1)_{\leq n})}{\mu_A((A_0)_{\leq n})} < 1 - \varepsilon$$

for relevant $0 < \varepsilon < 1$.

Again, using (10), obtain $\frac{\mu(Q \cap T_{n,k})}{\mu(T_{n,k})} \leq (1-\varepsilon)^t < (1-\varepsilon)^{t-1} < (1-\varepsilon)^{k/2-2}$ and therefore Q is μ -negligible relative to T for arbitrary choice of a direction function d . It is clear also that Q is exponentially μ -negligible relative to T with a $\delta = (1 - \varepsilon)^{1/2}$.

The proof of the last claim is analogous to the former one. •

2.3 Bidimensional Cesaro asymptotic density

Let $\{\mu_s, 0 < s < 1\}$ be a family of probabilistic distributions introduced in [3] for a free group $F(X)$ of finite rank. In terms of relative frequencies of R relative to F it can be written as follows

$$\mu_s(R) = s \sum_{k=0}^{\infty} f_k (1-s)^k.$$

In [3] it was also shown that the average (freely-reduced) length of words in $F(X)$, distributed according to μ_s is equal to $l = \frac{1}{s} - 1$; evidently, $l \rightarrow \infty$ while $s \rightarrow 0^+$. Therefore, the family $\{\mu_s\}$ can be parametrized by $l : \{\mu_l \mid 1 < l < \infty\}$.

Suppose $\mu_A = \{\mu_{A,l}\}$ and $\mu_B = \{\mu_{B,l}\}$, $1 < l < \infty$ are atomic probability measures on free groups A and B correspondingly. For asymptotic estimates of sets it is sufficient to assume that l runs over natural numbers. Let μ_l be the atomic probability measure on F induced by $\mu_{A,l}$ and $\mu_{B,l}$ and let $Q = A_1 * B_1 \subseteq T = A_0 * B_0 \subseteq F = A * B$. For some choice of direction function $d(l, k)$ consider a function $(l, k) \rightarrow \rho_{\mu_l}^{l,k}(Q, T)$ of relative frequencies.

Definition 2.3. Let $Q \subseteq T \subseteq F$ as above. The function

$$(l, k) \rightarrow \frac{\mu_l((F)^k \cap Q)}{\mu_l((F)^k \cap T)} = \rho_{\mu_l}^{l,k}(Q, T)$$

is called frequency function of Q relative to T . The limit (if it exists and does not depend on a choice of $d(l, k)$)

$$\rho^C(Q, T) = \lim_{d(l,k) \rightarrow \infty} \rho_{\mu_l}^{l,k}(Q, T),$$

is called the *bidimensional Cesaro asymptotic density* of Q relative to T .

If s is not small (for example, $s \geq \frac{1}{2}$), then every set containing 1 or short elements is not small since $\mu_s(1) = s$. We avoid this since l runs over natural l 's and $l > 1$.

We say that Q is C -negligible relative to T if $\rho^C(Q, T) = 0$; further, Q is *exponentially C -negligible* relative to T if $\rho_{\mu_l}^{l,k}(Q, T) < \delta^k$ for some constant $0 < \delta < 1$ and all sufficiently large k . Supplements to these sets called *C -generic* and *exponentially C -generic* correspondingly.

Here we describe some sufficient conditions on free product of sets to be exponentially negligible with respect to Cesaro asymptotic density, which we will use in Section 4 to evaluate sizes of subsets of normal forms.

Proposition 2.4. Let $F = A * B$ be a free product of two free groups A and B of finite ranks. Suppose $\mu_A = \{\mu_{A,l}\}$ and $\mu_B = \{\mu_{B,l}\}$, $1 < l < \infty$ are two families of atomic probability measures on A and B correspondingly and $\{\mu_l\}$ is induced family of measures on F . Let $A_1 \subseteq A_0 \subseteq A$, $B_1 \subseteq B_0 \subseteq B$ and densities $\rho_{\mu_{A,l}}(A_1, A_0)$, $\rho_{\mu_{B,l}}(B_1, B_0)$ exist. If there is a number $0 < q < 1$ such that $\frac{\mu_{A,l}(A_1)}{\mu_{A,l}(A_0)} < q$ or $\frac{\mu_{B,l}(B_1)}{\mu_{B,l}(B_0)} < q$ for all $l > l_0$ for some l_0 , then the set $Q = A_1 * B_1$ is exponentially C -negligible relative to $T = A_0 * B_0$.

Proof. Let us fix a pair of natural numbers (l, k) such that $l > l_0$. Suppose that $\frac{\mu_{A,l}(A_1)}{\mu_{A,l}(A_0)} < q$. Splitting T into layers as in Lemma 2.1, obtain

$$\rho_{\mu_l}^{l,k}(Q, T) = \begin{cases} \text{if } k = 2t & \frac{(\mu_{A,l}(A_1))^t (\mu_{B,l}(B_1))^t}{(\mu_{A,l}(A_0))^t (\mu_{B,l}(B_0))^t} \\ \text{if } k = 2t + 1 & \frac{(\mu_{A,l}(A_1))^t (\mu_{B,l}(B_1))^t (\mu_{A,l}(A_1) + \mu_{B,l}(B_1))}{(\mu_{A,l}(A_0))^t (\mu_{B,l}(B_0))^t (\mu_{A,l}(A_0) + \mu_{B,l}(B_0))}. \end{cases}$$

Therefore,

$$\rho_{\mu_l}^{k,l}(Q, T) \leq \left(\frac{\mu_{A,l}(A_1)}{\mu_{A,l}(A_0)} \right)^t \leq q^t$$

and since the limit does not depend on a choice of a direction, $A_1 * B_1$ is exponentially (with $\delta = q^{\frac{1}{2}}$) C -negligible relative to $A_0 * B_0$. •

The following lemma connects two types of measuring in free groups and will be very useful in the sequel:

Lemma 2.5. *Let A_1 be an exponentially negligible subset relative to a subset A_0 of a finitely generated free group $A = F(X)$. Then for arbitrary $l_0 > 1$ there is a real number $0 < q < 1$ such that for all $l > l_0$ holds*

$$\frac{\mu_{A,l}(A_1)}{\mu_{A,l}(A_0)} < q.$$

Proof. Since A_1 is exponentially negligible relative to A_0 there is a natural number n_0 and real number $0 < p < 1$ such that $f_n(A_1, A_0) = \frac{|(A_1)_n|}{|(A_0)_n|} < p^n$ for all $n \geq n_0$. In particular, $|(A_1)_n| < p^{n_0}|(A_0)_n|$ for all $n \geq n_0$. It is sufficient to show that for every fixed $s < s_0$ a number $\frac{\mu_{A,s}(A_1)}{\mu_{A,s}(A_0)}$ is bounded above by some positive constant $q < 1$.

By definition we have

$$\frac{\mu_{A,s}((A_1)_{\leq n})}{\mu_{A,s}((A_0)_{\leq n})} = \frac{s \sum_{k=0}^n |(A_1)_k| (1-s)^k}{s \sum_{k=0}^n |(A_0)_k| (1-s)^k} < \frac{\sum_{k=0}^n p^{n_0} |(A_0)_k| (1-s)^k}{\sum_{k=0}^n |(A_0)_k| (1-s)^k} < p^{n_0} \quad (11)$$

for all $n \geq n_0, s > s_0$. Without loss of generality one can assume that $n_0 > 1$, in opposite case add or remove arbitrary element of length 1 from A_1 . Passing to a limit in the inequality $\frac{\mu_{A,s}((A_1)_{\leq n})}{\mu_{A,s}((A_0)_{\leq n})} < p^{n_0}$, obtain $\frac{\mu_{A,s}(A_1)}{\mu_{A,s}(A_0)} \leq p^{n_0}$ and therefore $\frac{\mu_{A,s}(A_1)}{\mu_{A,s}(A_0)}$ is strictly bounded by positive constant $q = p^{n_0-1}$ for every $s < s_0$. •

3 Generation of random normal forms

Let $G = A *_C B$ be a free product of a free group A with a finite base X and a free group B with a finite base Y , amalgamated over a finitely generated subgroup C .

In the series of previous papers [3, 4] we described some algorithms in free groups of finite rank and amalgamated products of such groups. The

special role in analysis of computational complexity of algorithmic problem in group G play regular and stable normal forms. So, the goal of this section is a construction of procedures for generating of random reduced, random canonical normal and random cyclically reduced forms. The second goal is asymptotic estimation of sets of all regular, stable, singular and unstable forms relative to the set of all forms with the help of different asymptotic densities (see also in Section 4). We present below four generators of random normal forms.

3.1 Generator of reduced forms

The procedure RG_{rf} generates a random element in a reduced form of a syllable length k . This procedure depends on a given probability distribution $\theta : \mathbb{N} \rightarrow \mathbb{R}^+$ on the set of natural numbers \mathbb{N} with zero, two fixed probability distributions μ_A and μ_B on $A \setminus C$ and $B \setminus C$, and two probability distributions $\mu_{A,C}, \mu_{B,C}$ on C , where C is viewed as a subgroup of A or B correspondingly.

Procedure 3.1. (*Generator RG_{rf} of a random element in the reduced form (5)*)

INPUT: Number k chosen with respect to a fixed probability distribution $\theta : \mathbb{N} \rightarrow \mathbb{R}^+$.

OUTPUT: A random word u in the reduced form of length k .

COMPUTATIONS:

- 1) Choose A or B with equal probability $\frac{1}{2}$.
- 2) If $k = 0$ then
 - a) if the choice in 1) is A then choose randomly an element c in C with probability $\mu_{A,C}$;
 - b) if the choice in 1) is B then choose randomly an element c in C with probability $\mu_{B,C}$.

Output $u = c$.

- 3) If $k > 0$ then do the following
 - a) if the choice in 1) is A then choose $g_1 \in A \setminus C$ with probability μ_A , then an element $g_2 \in B \setminus C$ with probability μ_B , and repeat this process choosing alternatively $g_i \in A \setminus C$ and $g_{i+1} \in B \setminus C$ until k elements g_1, \dots, g_k are constructed.

- b) if the choice in 1) is B then choose $g_1 \in B \setminus C$ with probability μ_B , then an element $g_2 \in A \setminus C$ with probability μ_A , and repeat this process as in step 3.a).

Output $u = g_1 \dots g_k$.

Remark 3.1. Generator RG_{ef} of a random element g in the freely reduced form (2) can be constructed in the similar way. Namely, we should take elements from A and B consequently to get a result. We will use this generator later in Section 4.

3.2 Generator of canonical normal forms

Let $G = A *_C B$, and suppose S, T are fixed Schreier transversals for C in A and B respectively.

Denote by RG_{cnf} the following procedure for generating of random elements in the canonical normal form of syllable length k . This procedure depends on a given probability distribution $\theta : \mathbb{N} \rightarrow \mathbb{R}^+$, two fixed probability distributions μ_A and μ_B on $A \setminus C$ and $B \setminus C$, and two probability distributions $\mu_{A,C}, \mu_{B,C}$ on C .

Procedure 3.2. (*Generator RG_{cnf} of a random element in the canonical normal form (3)*)

INPUT: A natural number k chosen with respect to a fixed probability distribution $\theta : \mathbb{N} \rightarrow \mathbb{R}^+$.

OUTPUT: A random word v in the canonical normal form of length k .

COMPUTATIONS:

- 1) Choose A or B with equal probability $\frac{1}{2}$ and do as in the Procedure 3.1:
 - a) if the choice in 1) is A then choose randomly an element c in C with probability $\mu_{A,C}$;
 - b) if the choice in 1) is B then choose randomly an element c in C with probability $\mu_{B,C}$;
- 2) If $k = 0$ then output $v = c$.
- 3) If $k \geq 1$ and

- a) the choice in 1) is A then choose $g_1 \in A \setminus C$, represent it as $g_1 = c_1 s_1$, where $c_1 \in C$, $s_1 \in S$ (so, $\mu_A(Cg_1) = \mu_A(Cs_1)$) and repeat this choosing alternatively $g_i \in A \setminus C$ and $g_{i+1} \in B \setminus C$ with probabilities $\mu_A(Cg_i)$, $\mu_B(Cg_{i+1})$ and represent $g_i = c_i s_i$, $g_{i+1} = c_{i+1} t_{i+1}$ until k elements $s_1, t_2, s_3, t_4, \dots$ are constructed.

Output $v = cs_1 t_2 s_3 t_4 \dots$

- b) the choice in 1) is B then choose $g_1 \in B \setminus C$, represent it in a form $g_1 = c_1 t_1$, where $c_1 \in C$, $t_1 \in T$ and repeat this procedure as in 3a).

Output $v = ct_1 s_2 t_3 s_4 \dots$

Remark 3.2. Two probability distributions μ_A and μ_B on $A \setminus C$ and $B \setminus C$ we describe in details in Section 4.1.

Now we construct one more generator.

3.3 Generator of cyclically reduced normal forms

Let RG_{crf} be the following procedure for generating of random elements in the cyclically reduced canonical forms of length $2k$ or 1. This procedure depends on a given probability distribution $\theta : 2\mathbb{N} \cup \{1\} \rightarrow \mathbb{R}^+$, two fixed probability distributions μ_A and μ_B on $A \setminus C_A^*$ and $B \setminus C_B^*$, and two probability distributions $\mu_{A,C}, \mu_{B,C}$ on C , where $C_A^* = \bigcup_{x \in A} C^x$ and $C_B^* = \bigcup_{x \in B} C^x$.

Procedure 3.3. (*Generator RG_{crf} of a random element in the cyclically reduced normal form (6)*)

INPUT: An even natural number k or 1 chosen with respect to θ .

OUTPUT: A random word w in the cyclically reduced normal form of length k .

COMPUTATIONS:

- 1) Choose A or B with equal probability $\frac{1}{2}$ and do as in Procedure 3.1:
 - a) if the choice in 1) is A then choose randomly an element c in C with respect to probability $\mu_{A,C}$;
 - b) if the choice in 1) is B then choose randomly an element c in C with respect to probability $\mu_{B,C}$;

- 2) If $k = 0$ then output $w = c$.
- 3) If $k = 1$
 - a) if the choice in 1) is A then choose randomly an element $g_1 \in A \setminus C_A^*$ with the probability $\mu_{A \setminus C_A^*}(Cg_1)$, represent it as $g_1 = c_1 s_1$, where $c_1 \in C$, $s_1 \in S$ and output $w = c s_1$.
 - b) if the choice in 1) is B then choose randomly an element $g_1 \in B \setminus C_B^*$ with the probability $\mu_{B \setminus C_B^*}(Cg_1)$, represent it as $g_1 = c_1 t_1$, where $c \in C$, $t_1 \in T$ and output $w = c t_1$.
- 4) If $k = 2l$, $l \geq 1$ then do the following
 - a) if the choice in 1) is A then choose $g_1 \in A \setminus C^*$ as in 3), then an element $g_2 \in B \setminus C^*$ with probability $\mu_B(Cg_2)$ and represent $g_2 = c_2 t_2$, and repeat this process choosing alternatively $g_i \in A \setminus C^*$ and $g_{i+1} \in B \setminus C^*$ with probabilities $\mu_A(Cg_i)$, $\mu_B(Cg_{i+1})$ and represent $g_i = c_i s_i$, $g_{i+1} = c_{i+1} t_{i+1}$ until k elements $s_1, t_2, s_3, t_4, \dots$ are constructed.
Output $w = c s_1 t_2 s_3 t_4 \dots$
 - b) if the choice in 1) is B then choose $g_1 \in B \setminus C^*$ as in 3), then an element $g_2 \in A \setminus C^*$ with probability $\mu_A(Cg_2)$ and represent $g_2 = c_2 s_2$, and repeat as in 4a).
Output $w = c t_1 s_2 t_3 s_4 \dots$

Remark 3.3. We shall talk about probability distributions on cosets $\mu_A(Cg_i)$, $\mu_B(Cg_i)$ and other sets later in Section 4.

4 Evaluation of sets of randomly generated normal forms

Let $G = A *_C B$, where A and B are free groups with finite bases X and Y correspondingly, and C is a finitely generated subgroup. Now we are ready to stratify sets of generated random normal forms of elements in G into regular, singular and stable or unstable subsets and evaluate their sizes in whole sets of corresponding forms.

4.1 Atomic measures of elements in normal forms

In this section we introduce probability measures on sets of normal forms, constructed with a help of generators above.

The probability to obtain an element $g \in G$ in freely reduced or reduced form of syllable length k on the output of generator RG_{ef} or RG_{rf} is equal to

$$\mu_k(g) = \frac{1}{2} \mu_1(g_1) \cdots \mu_k(g_k),$$

where $\mu_i(g_i) = \mu_A(g_i)$ if $g_i \in A$, and $\mu_i(g_i) = \mu_B(g_i)$ otherwise, $i = 1, \dots, k$; $k \geq 1$; and if $k = 0$ then $\mu_0(c) = \mu_C(c)$, where $\mu_C(c) = \mu_{A,C}$ if C is viewed as a subgroup of A , and $\mu_C(c) = \mu_{B,C}$ otherwise. Clearly, μ_k is an atomic probability measure on sets \mathcal{EF}_k or \mathcal{RF}_k of all (freely) reduced elements of length k . Now one can calculate a probability measure μ on sets $\mathcal{EF}, \mathcal{RF}$ of all (freely) reduced elements:

$$\mu(g) = \theta(k) \mu_k(g).$$

To complete definitions above we have to define probability measures $\mu_A, \mu_B, \mu_{A,C}$, and $\mu_{B,C}$. There are a lot of different methods to describe a probability distribution on a free group; for example, it can be done as in [7] with a help of no-return random walk W_s ($s \in (0, 1]$) on the Cayley graph of $A = F(X)$ of rank $r = |X|$. The probability $\mu_s(g)$ for this process to terminate at g is given by the formula

$$\mu_s(g) = \frac{s(1-s)^{|g|}}{2m \cdot (2m-1)^{|g|-1}} \quad \text{for } w \neq 1 \quad (12)$$

and

$$\mu_s(1) = s. \quad (13)$$

This random walk can be considered on the set $A \setminus C$ with small changes. Since the set $A \setminus C$ is regular in $A = F(X)$ there is a simple procedure to define probability measures using random walks in the corresponding finite automata (or graph). We introduce these measures using different way, i.e. infinite trees, the Cayley graphs of A and B . Let Γ be the Cayley graph of A with respect to a basis X of A , Γ_C is a subgroup graph for C , and Γ_C^* its extended graph. Denote by $\pi : \Gamma \rightarrow \Gamma_C^*$ the unique canonical projection from Γ onto Γ_C^* , so π is a morphism of graphs which preserves labels. We choose a real number $s \in (0, 1)$ and define a no-return random walk \mathcal{W}_s on

Γ as follows. If \mathcal{W}_s is at vertex $v \in \Gamma$ and $v \notin V(\Gamma_C)$ then \mathcal{W}_s stops at v with probability s and moves from v (away from the root 1_C) along an adjacent edge with equal probability $\frac{1-s}{2r-1}$. If \mathcal{W}_s is at vertex $v \in V(\Gamma_C)$ then \mathcal{W}_s moves from v (away from the root 1_C) along any adjacent edge with equal probability $\frac{1}{2r-1}$. The random walk \mathcal{W}_s induces (via the projection π) a random walk \mathcal{W}_s^* on Γ_C^* . Then the probability $\mu_s(w)$ for \mathcal{W}_s to stop at $w \in A \setminus C$ can be written as follows

$$\mu_s(w) = \frac{1}{2r} \frac{(1-s)^{|w|-m_w}}{(2r-1)^{|w|-m_w}} \frac{1}{(2r-1)^{m_w-1}},$$

where $|w|$ is a freely-reduced length of w in A and m_w is the number of times the path $\pi(w)$ visits the vertex 1_C in Γ_C^* .

Now we calculate probability to obtain an element in the canonical and cyclically reduced normal forms. Namely, the probability to obtain $v = cp_1p_2 \dots p_k$, $k \geq 0$, is equal to

$$\mu_k(v) = \frac{1}{2} \mu_C(c) \mu_1(p_1) \mu_2(p_2) \dots \mu_k(p_k),$$

where $\mu_i(p_i) = \mu_{F(p_i)}(Cp_i)$, $i = 0, \dots, k$ and $\mu_C(c) = \mu_{A,C}$ if C is viewed as a subgroup of A , and $\mu_C(c) = \mu_{B,C}$ otherwise.

Clearly, μ_k is an atomic probability measure on the set \mathcal{CNF}_k of all canonical normal (or \mathcal{CRF}_k of cyclically reduced) forms of syllable length k . Then the probability measure μ on sets $\mathcal{CNF}, \mathcal{CRF}$ of all canonical (or cyclically reduced) normal forms is equal to

$$\mu(v) = \theta(k) \mu_k(v).$$

To complete this definition, we have to know how to calculate probability measures of subgroup C and its cosets. It can be done with a help of consolidated subgroup graph for C (see Section 3.3 of [3] for details).

4.2 Measures on sets of normal forms

In the rest of the paper by $\mathcal{NF}_r(\mathcal{NF}_s)$ we denote the set of all regular (stable) elements in normal form (prefix \mathcal{NF} will changes depends on type of chosen form) and by $\mathcal{NF}_{\text{sin}}(\mathcal{NF}_{\text{uns}})$ its complement in \mathcal{NF} .

Decision of algorithmic problems and problem of stratification of inputs leads to necessity of estimation of sizes of normal forms and their regular and

stable subsets. For this purpose we use two following approaches: asymptotic approach and related notion of L -measure and probabilistic approach and the notion of λ_L -measure. To apply classification theorems about regular sets, we describe in this section structure of \mathcal{NF} and $\mathcal{NF}_{\text{uns}}$.

Remind some definitions first. In [7] for subsets R, L of a free group F of a finite rank was defined their *size ratio* at length n by

$$f_n(R, L) = \frac{f_n(R)}{f_n(L)} = \frac{|R \cap S_n|}{|L \cap S_n|}.$$

The *asymptotic density* of R relative to L is defined by

$$\rho(R, L) = \overline{\lim}_{n \rightarrow \infty} f_n(R, L).$$

By $r_L(R)$ we denote the *cumulative size ratio* of R relative to L :

$$r_L(R) = \sum_{n=1}^{\infty} f_n(R, L).$$

R is called *negligible* relative to L if $\rho(R, L) = 0$. A set R is termed *exponentially negligible* relative to L (or *exponentially L -negligible*) if $f_n(R, L) \leq \delta^n$ for all sufficiently large n and some positive constant $\delta < 1$. Further, a set R is called *generic* relative to L if $\lim_{n \rightarrow \infty} f_n(R, L)$ exists and equal to 1; R is termed *exponentially generic* relative to L if there exists a positive constant $\delta < 1$ such that $1 - \delta^n < f_n(R, L) < 1$ for large enough n . We will use in this section the following notion of a thick set; namely, a set R is *thick* relative to L if $\lim_{n \rightarrow \infty} f_n(R, L)$ exists and strictly greater than 0.

In [7] were also introduced notions of λ_L -measurable and exponentially λ_L -measurable set R . This way of measuring closed to the frequency measure and also coming up from no-return non-stop random walk on an automaton for L . Namely,

$$\lambda_L(R) = \sum_{w \in R} \lambda_L(w) = \sum_{n=0}^{\infty} f'_n(R, L),$$

where

$$f'_n(R, L) = \sum_{w \in R \cap S_n} \lambda_L(w),$$

and $\lambda_L(w)$, $f'_n(R, L)$ defined by a random walk on an automaton for L (see Section 5.2, p. 109 of [7] for details). Note, that generally speaking, $f'_n(R, L)$ differs from $f_n(R, L)$ defined above because of specific of an automaton for L . On the other hand, $f'_n(R, F(X)) = f_n(R, F(X))$. Notice also, that λ_L is multiplicative, since $\lambda_L(uv) = \lambda_L(u)\lambda_L(v)$ for any $u, v \in R$ such that $uv = u \circ v$ and $uv \in R$; it is easy to check by definition of random walk on L .

We say that R is λ_L -measurable, if $\lambda_L(R)$ is finite. A set R is termed *exponentially λ_L -measurable*, if $f'_n(R, L) \leq q^n$ for all sufficiently large n and some positive constant $\delta < 1$.

In [7] was given the following definition. For every $w \in F$ the set $C_L(w) = L \cap C(w)$ is called an L -cone, and $C_L(w)$ is called L -small, if it is exponentially λ_L -measurable.

Theorem 4.1. [7, Theorem 5.4]. *Let R be a regular subset of a prefix-closed regular set L in a finite rank free group F . Then either the prefix closure \overline{R} of R in L contains a non-small L -cone or \overline{R} is exponentially λ_L -measurable.*

This theorem is very convenient for asymptotic estimates of subsets of F and we are going to apply it in Theorem B. To prove this theorem we will use also lemmata 4.2 and 4.4:

Lemma 4.2. *Suppose $F = F(X)$ is a finite rank free group and $W = \{w_1, \dots, w_d\}$ is a finite subset of F such that at least one w_i is not trivial. Then*

$$F_0 = \{f \in F \mid f \text{ does not contain any element of } W \text{ as a subword} \}$$

is a regular, prefix-closed exponentially λ_F -measurable set.

Proof. Remind Myhill-Nerode criterion, which gives necessary and sufficient conditions on a subset of semigroup to be regular. For a language R in semigroup A^* consider an equivalence relation \sim_R in A^* relative to R : $v_1 \sim_R v_2$ if and only if for each string u over A the words v_1u and v_2u are either simultaneously in R or not in R .

Myhill-Nerode. A set R is regular in A^* iff there are only finitely many \sim_R -equivalence classes.

Proof of this theorem can be found, for example, in [6] (see Theorem 1.2.9).

Since we work in a free group, the analogue of latter theorem for groups was proved in [7]. Now, let $R \subseteq F$. Define an equivalence relation \sim on F relative to R such that $v_1 \underset{R}{\sim} v_2$ if and only if for each $u \in F$ the following condition holds: $v_1 u = v_1 \circ u$ and $v_1 u \in R$ if and only if $v_2 u = v_2 \circ u$ and $v_2 u \in R$.

Lemma 4.3. [7, Lemma 5.3]. *Let $R \subseteq F$. Then R is regular if and only if there are only finitely many $\underset{R}{\sim}$ -equivalence classes.*

It is sufficient to show the statement of lemma 4.2 for a singleton set W . Indeed, let F_0 be the subset of words in F that do not contain w as a subword, and F_1 be the set of words that do not contain w, w_1, \dots, w_d . Then $F_1 \subseteq F_0$, and F_1 is exponentially λ_F -measurable if F_0 is so. Assume now that $W = \{w\}$ and $|w| = t \geq 1$.

We describe all equivalence classes relative to F_0 :

- 1) $\mathcal{K}_0 = F \setminus F_0$;
- 2) for all $u \in S_{t-1}$ define $\mathcal{K}_u = \{f \in F : f = g \circ u, g \in F, u \in S_{t-1}\} \setminus \mathcal{K}_0$;
- 3) for all $v \in B_{t-2}$ set $\mathcal{K}_v = \{v\}$.

Now we will prove that this decomposition defines an equivalence relation on F relative to F_0 , i.e. for every representative $u_1, u_2 \in F$ holds $u_1 \underset{F_0}{\sim} u_2 \Leftrightarrow$ (for all $p \in F : u_1 p = u_1 \circ p$ and $u_1 p \in F_0$ iff $u_2 p = u_2 \circ p$ and $u_2 p \in F_0$).

Obviously, this condition holds for classes \mathcal{K}_0 and \mathcal{K}_v from i.3. We shall show that it holds also for classes \mathcal{K}_u . Let $u_1, u_2 \in \mathcal{K}_u$, where $u \in S_{t-1}$, i.e. elements $u_1 = g_1 \circ u, u_2 = g_2 \circ u$ and u_1, u_2 do not contain w as a subword. If $u_1 p = (g_1 \circ u) \circ p \in F_0$, then $u_2 p = g_2 \circ u \circ p$ by definition of class. So we should show also, that $g_2 \circ u \circ p$ doesn't belong to \mathcal{K}_0 . But since neither $g_2 \circ u$ nor $u \circ p$ contains w as a subword, then w should have a form $w = g'_2 \circ u \circ p'$, where nontrivial element g'_2 is an end of g_2 and p' is nontrivial beginning of p . But it implies that $t = |w| = |g'_2| + |u| + |p'| > t$, a contradiction. Suppose now that $u_1 p = (g_1 \circ u) \circ p \notin F_0$. Then exactly $u \circ p$ contains w as a subword. Hence, the element $u_2 p = g_2 \circ u \circ p$ also contains w as a subword.

Therefore, $\underset{F_0}{\sim}$ is a relation equivalence on F relative to F_0 , and since the number of equivalence classes relative to F_0 is finite, it follows from lemma

4.3 that F_0 is regular in F . By definition of F_0 it is prefix-closed in F . Further, since F_0 evidently doesn't contain a cone, then by Theorem 4.1 F_0 is exponentially λ_F -measurable. The last statement about exponential measurability of F_0 in F follows also from Lemma 3 in [2]. •

Lemma 4.4. *Suppose $F = F(X)$ is a finite rank free group and $W = \{w_1, \dots, w_d\}$ is a finite subset of F , at least one of w_i is not trivial and F_0 is a set of words that do not contain elements of W as subwords. Let F_1 denote the supplement of F_0 in F . Let $R \subseteq F_0$ be regular and $L \subseteq F$ be regular prefix-closed in F . If for every non-small cone $C_L(u)$ in L holds $C_L(u) \cap F_1 \neq \emptyset$, then R is exponentially λ_L -measurable.*

Proof. By assumption of lemma the set $R \cap F_1$ is nonempty, but all non-small cones $C_L(u)$ have nonempty intersecetion with F_1 ; therefore, $C_L(u) \not\subseteq R$ and by theorem 4.1 the set R is exponentially λ_L -measurable. •

To estimate sizes of subsets of normal forms we prove first the following theorem which describes their structure.

Theorem 4.5. *Let $G = A \underset{C}{*} B$ be an amalgamated product, where A, B, C are free groups of finite rank. Then the set $\mathcal{NF}_{\text{uns}}$ is regular in G and the set \mathcal{NF} is regular prefix closed in G for all $\mathcal{NF} = \{\mathcal{EF}, \mathcal{RF}, \mathcal{CNF}, \mathcal{CRF}\}$.*

Proof. Suppose L, M are two regular sets in alphabets $X \cup X^{-1}, Y \cup Y^{-1}$ correspondingly (recall that $A = F(X), B = F(Y)$). Denote by LM concatenation of sets L and M and by L^* a monoid generated by L . We will often use the following formula in the sequel:

$$L * M = L \sqcup M \sqcup LML \sqcup MLM \sqcup L(ML)^*M \sqcup M(LM)^*L \quad (14)$$

Particulary, it shows that for regular L, M the set $L * M$ is also regular (in alphabet $X \cup X^{-1} \cup Y \cup Y^{-1}$).

Consider the set of all freely reduced normal forms:

$$\mathcal{EF} = A * B. \quad (15)$$

Evidently, the set \mathcal{EF} is regular and prefix closed.

The set of all unstable freely reduced normal forms:

$$\mathcal{EF}_{\text{uns}} = \left(\bigcup_{s \in S_{\text{uns}}} Cs \right) * \left(\bigcup_{t \in T_{\text{uns}}} Ct \right). \quad (16)$$

By Lemma 1.3 the set S_{uns} is a finite union of left cosets of C of the type $s_1 s_2^{-1} C$, where $s_1, s_2 \in S_{\text{int}}$. The set $\mathcal{EF}_{\text{uns}}$ is regular as a free product of regular sets (see formula (14)) since $\bigcup_{s \in S_{\text{uns}}} Cs$ (as well as $\bigcup_{t \in T_{\text{uns}}} Ct$) is a concatenation of regular sets C and S_{uns} (or T_{uns}).

Consider all non-trivial reduced forms:

$$\mathcal{RF} = (A \setminus C) * (B \setminus C) \quad (17)$$

The set of all unstable reduced forms:

$$\mathcal{RF}_{\text{uns}} = (A'_1 \setminus C) * (B'_1 \setminus C) \quad (18)$$

such that

$$A'_1 = \bigcup_{s \in S_{\text{uns}}} Cs \quad \text{and} \quad B'_1 = \bigcup_{t \in T_{\text{uns}}} Ct.$$

Since difference of regular sets is regular again, both sets of forms are evidently regular. To see that $A \setminus C$ is a prefix closed in A set, one can identify it with a language of all words in a Schreier graph $\Gamma^* = \Gamma_C^*$ readable as a labels of paths starting in a root vertex 1_C (and probably return to this vertex again) but finish in arbitrary vertex of Γ^* except the root one (remind that the graph Γ^* was defined in Section 1.2). Clearly, such a language is prefix closed in A . Analogously, $B \setminus C$ is prefix closed in B and so \mathcal{RF} is prefix closed as a free product of prefix closed sets.

Let \mathcal{NF} be the set of all canonical normal forms. Every $v \in \mathcal{CNF}$ can be written in the form (3), i.e. as

$$v = cp_1 p_2 \dots p_l,$$

where $c \in C$, $p_i \in (S \cup T) \setminus 1$, and $F(p_i) \neq F(p_{i+1})$, $i = 1, \dots, l, l \geq 0$. Now consider the set of all canonical normal forms:

$$\mathcal{CNF} = C \circ_t (S * T) \quad (19)$$

At the same time,

$$\mathcal{CNF} = A \circ (S * T)_1 \sqcup B \circ (S * T)_2, \quad (20)$$

where $(S * T)_1$ starts from $t \in T$ and $(S * T)_2$ starts from $s \in S$. Substitute \circ with concatenation and applying formula (14) for $(S * T)_i, i = 1, 2$, conclude

that \mathcal{CNF} is regular set. Since A , S , and T are prefixed closed, the set \mathcal{CNF} also has this property.

Analogous decomposition can be applied to $\mathcal{CNF}_{\text{uns}}$:

$$\mathcal{CNF}_{\text{uns}} = C \circ_t (S_{\text{uns}} * T_{\text{uns}}) = A'_1 \circ (S_{\text{uns}} * T_{\text{uns}})_1 \sqcup B'_1 \circ (S_{\text{uns}} * T_{\text{uns}})_2, \quad (21)$$

where A'_1, B'_1 obtained as intersections of two regular sets and $(S_{\text{uns}})_i, (T_{\text{uns}})_i, i = 1, 2$ are regular by Proposition 1.3.

The proof of this theorem for \mathcal{CRF} is straightforward. •

4.3 Regular and stable normal forms

Notions of regular and stable forms was formulated in Section 1.2, and now we are ready to prove the main theorems about evaluation of the set of singular and unstable (and, therefore, regular and stable) normal forms in group G .

Theorem A. *Let $G = A *_C B$ be an amalgamated product, where A, B, C are free groups of finite rank. Then for every set of normal forms $\mathcal{NF} = \{\mathcal{EF}, \mathcal{RF}, \mathcal{CNF}, \mathcal{CRF}\}$*

- (i) *If C has a finite index in A and in B , then every normal form is singular and unstable, i.e. $\mathcal{NF}_{\text{sin}} = \mathcal{NF}_{\text{uns}} = \mathcal{NF}$;*
- (ii) *If C of infinite index either in A or in B , then \mathcal{NF}_r and \mathcal{NF}_s are exponentially μ -generic relative to \mathcal{NF} , and $\mathcal{NF}_{\text{sin}}$ and $\mathcal{NF}_{\text{uns}}$ are exponentially μ -negligible relative to \mathcal{NF} in the following cases:*
 - (ii.1) *μ is defined by pseudo-measures μ_A and μ_B , which are cardinality functions on A and B correspondingly; in this case ρ_μ is a bidimensional asymptotic density;*
 - (ii.2) *μ is defined by atomic probability measures $\mu_{A,l}$ and $\mu_{B,l}$ on A and B correspondingly; in this case ρ^C is a bidimensional Cesaro asymptotic density.*

Proof. Observe, that all singular representatives by Proposition 1.3 are also unstable, and it will be sufficient to prove Theorem A for unstable normal forms only.

Suppose first that C has a finite index in both A and B . Then there are nontrivial subgroups N_A in C , which is normal subgroup of A , and N_B in C , which is normal in B . Therefore, $A * B = N_{A*B}^*(C)$. Then by definition all cosets representatives (and hence \mathcal{NF}) are unstable, i.e. $\mathcal{NF}_{\text{sin}} = \mathcal{NF}_{\text{uns}} = \mathcal{NF}$.

Suppose now, that subgroup C has an infinite index in A . Let $\mathcal{NF} = \{\mathcal{EF}, \mathcal{RF}\}$ and (ii.1) is hold. By Proposition 2.2 it is sufficient to show that $\mathcal{NF} = A_0 * B_0$, $\mathcal{NF}_{\text{uns}} = A_1 * B_1$ and A_1 is exponentially negligible relative to A_0 (it is clear that the density $\rho_{\mu_B}(B_1, B_0)$ for these forms exists because of definitions of B_0 and B_1).

Consider freely reduced forms first. Applying formulae (15), (16), set $A_0 = A; B_0 = B$ and $A_1 = \bigcup_{s \in S_{\text{uns}}} Cs; B_1 = \bigcup_{t \in T_{\text{uns}}} Ct$. Observe, that S_{uns} is exponentially negligible in S by Corollary 1.4 and so A_1 is exponentially negligible in A_0 by Proposition 4.7. from [7].

For the set of all non-trivial reduced forms, using (17), (18), set $A_0 = A \setminus C; B_0 = B \setminus C$ and $A_1 = A'_1 \setminus C; B_1 = B'_1 \setminus C$, where $A'_1 = \bigcup_{s \in S_{\text{uns}}} Cs$ and $B'_1 = \bigcup_{t \in T_{\text{uns}}} Ct$. We have already shown for freely reduced forms, that A'_1 is exponentially negligible in A , and hence there is a q , $0 < q < 1$, such that $\frac{\mu_A((A'_1)_n)}{\mu_A((A)_n)} < q$ for all $n \geq n_0$ for some natural number n_0 .

It is clear that $\frac{\mu_A((C)_n)}{\mu_A((A)_n)} < q$, where $n \geq n_0$. Then

$$\begin{aligned} \mu_A((A_1)_n) &= \mu_A(((A'_1)_n) \setminus ((C)_n)) = \\ &= \mu_A((A'_1)_n) - \mu_A((C)_n) < q\mu_A((A)_n) - \mu_A((C)_n) < \\ &< q(\mu_A((A)_n) - \mu_A((C)_n)). \end{aligned}$$

Then we obtain $\frac{\mu_A((A_1)_n)}{\mu_A((A_0)_n)} < q$ for all $n \geq n_0$ and so the set A_1 is exponentially negligible relative to A_0 .

Suppose now that the case (ii.2) holds, i.e. the measure μ is defined on $F = A * B$ by atomic probability measures $\mu_{A,l}$ and $\mu_{B,l}$ on A and on B correspondingly and ρ^C is a bidimensional Cesaro asymptotic density. Since $A'_1 = \bigcup_{s \in S_{\text{uns}}} Cs$ is exponentially negligible in A , by Lemma 2.5 there exists a

number q , where $0 < q < 1$ such that $\frac{\mu_{A,l}(A'_1)}{\mu_{A,l}(A)} < q$ for all $l \geq l_0$. Then by

Proposition 2.4 the set of all unstable freely reduced forms is C -negligible relative to the set \mathcal{EF} .

Further, since $\mu_{A,l}(A'_1) - \mu_{A,l}(C) < q\mu_{A,l}(A) - \mu_{A,l}(C) < q(\mu_{A,l}(A) - \mu_{A,l}(C))$ for all $l \geq l_0$, obtain

$$\frac{\mu_{A,l}(A'_1 \setminus C)}{\mu_{A,l}(A \setminus C)} < q.$$

Therefore, by Proposition 2.4 we obtain claim (ii.2) for \mathcal{RF} .

Now consider canonical normal forms, given by formulae (20) and (21):

$$\mathcal{CNF} = (A \circ (S * T)_1) \sqcup (B \circ (S * T)_2) \text{ and } \mathcal{CNF}_{\text{uns}} = (A \circ (S_{\text{uns}} * T_{\text{uns}})_1) \sqcup (B \circ (S_{\text{uns}} * T_{\text{uns}})_2).$$

Clearly, it is enough to show the statement of the theorem for a pair $\Sigma = (A \circ (S * T)_1)$ and $\Sigma_{\text{uns}} = (A \circ (S_{\text{uns}} * T_{\text{uns}})_1)$. Denote by \hat{A} the set $A \setminus \{1\}$.

We shall show that unstable canonical normal forms are μ -exponentially negligible in the set of canonical normal forms relative to bidimensional asymptotic density defined by cardinality functions. By definition of a frequency function

$$\begin{aligned} \rho_{\mu}^{n,k}(\Sigma_{\text{uns}}, \Sigma) &= \frac{\mu(\Sigma_{\text{uns}} \cap \Sigma_{n,k})}{\mu(\Sigma_{n,k})} = \\ &= \frac{\mu_A((\hat{A})_{\leq n})\mu(((S_{\text{uns}} * T_{\text{uns}})_1)_{n,k-1}) + \mu(((S_{\text{uns}} * T_{\text{uns}})_1)_{n,k})}{\mu_A((\hat{A})_{\leq n})\mu(((S * T)_1)_{n,k-1}) + \mu(((S * T)_1)_{n,k})} = \\ &= \frac{\mu_A((\hat{A})_{\leq n})\mu(((S_{\text{uns}} * T_{\text{uns}})_1)_{n,k-1}) + \mu(((S_{\text{uns}} * T_{\text{uns}})_1)_{n,k})}{\mu_A((\hat{A})_{\leq n})\mu(((S * T)_1)_{n,k-1}) + \mu(((S * T)_1)_{n,k})} = \\ &= \frac{\mu_A((\hat{A})_{\leq n}) \cdot (\mu_B((T_{\text{uns}})_{\leq n}))^{[\frac{k}{2}]} (\mu_A((S_{\text{uns}})_{\leq n}))^{[\frac{k-1}{2}]} + (\mu_B((T_{\text{uns}})_{\leq n}))^{[\frac{k+1}{2}]} (\mu_A((S_{\text{uns}})_{\leq n}))^{[\frac{k}{2}]}}{\mu_A((\hat{A})_{\leq n}) \cdot (\mu_B((T)_{\leq n}))^{[\frac{k}{2}]} (\mu_A((S)_{\leq n}))^{[\frac{k-1}{2}]} + (\mu_B((T)_{\leq n}))^{[\frac{k+1}{2}]} (\mu_A((S)_{\leq n}))^{[\frac{k}{2}]}}. \end{aligned}$$

Suppose k is odd. Therefore, $\rho_{\mu}^{n,k}(\Sigma_{\text{uns}}, \Sigma) =$

$$\frac{(\mu_A((\hat{A})_{\leq n}) + \mu_B((T_{\text{uns}})_{\leq n})) (\mu_B((T_{\text{uns}})_{\leq n}))^{\frac{k-1}{2}} (\mu_A((S_{\text{uns}})_{\leq n}))^{\frac{k-1}{2}}}{(\mu_A((\hat{A})_{\leq n}) + \mu_B((T)_{\leq n})) (\mu_B((T)_{\leq n}))^{\frac{k-1}{2}} (\mu_A((S)_{\leq n}))^{\frac{k-1}{2}}}.$$

Since set S_{uns} is also exponentially negligible relative to S , $\rho_{\mu_A}(S_{\text{uns}}, S) <$

1. Then

$$\lim_{n \rightarrow \infty} \frac{\mu_A((S_{\text{uns}})_n)}{\mu_A((S)_n)} = \lim_{n \rightarrow \infty} \frac{\mu_A((S_{\text{uns}})_{\leq n})}{\mu_A((S)_{\leq n})} \text{ and } \frac{\mu_A((S_{\text{uns}})_{\leq n})}{\mu_A((S)_{\leq n})} < (1 - \varepsilon_1).$$

By the same reason $\frac{\mu_B((T_{\text{uns}})_{\leq n})}{\mu_B((T)_{\leq n})} < (1 - \varepsilon_2)$ for some $0 < \varepsilon_1, \varepsilon_2 \leq 1$ and

large enough n .

Thereby,

$$\rho_{\mu}^{n,k}(\Sigma_{\text{uns}}, \Sigma) < \left(\frac{\mu_A((\hat{A})_{\leq n}) + \mu_B((T_{\text{uns}})_{\leq n})}{\mu_A((\hat{A})_{\leq n}) + \mu_B((T)_{\leq n})} \right) ((1 - \varepsilon_1)(1 - \varepsilon_2))^{\frac{k-1}{2}}$$

for large enough n and odd k .

One can check that for all even k frequencies $\rho_{\mu}^{n,k}(\Sigma_{\text{uns}}, \Sigma)$ are bounded

$$\text{by } \left(\frac{\mu_A((\hat{A})_{\leq n}) + \mu_A((S_{\text{uns}})_{\leq n})}{\mu_A((\hat{A})_{\leq n}) + \mu_A((S)_{\leq n})} \right) (1 - \varepsilon_2) ((1 - \varepsilon_1)(1 - \varepsilon_2))^{\frac{k-2}{2}}.$$

$$\text{Fractions } \left(\frac{\mu_A((\hat{A})_{\leq n}) + \mu_B((T_{\text{uns}})_{\leq n})}{\mu_A((\hat{A})_{\leq n}) + \mu_B((T)_{\leq n})} \right) \text{ and } \left(\frac{\mu_A((\hat{A})_{\leq n}) + \mu_A((S_{\text{uns}})_{\leq n})}{\mu_A((\hat{A})_{\leq n}) + \mu_A((S)_{\leq n})} \right)$$

do not depend on k and less than 1 for all large enough n .

Let $\varepsilon = \min \{\varepsilon_1, \varepsilon_2\}$. Then

$$\rho_{\mu}^{n,k}(\Sigma_{\text{uns}}, \Sigma) < \left(\frac{\mu_A((\hat{A})_{\leq n}) + \mu_B((T_{\text{uns}})_{\leq n})}{\mu_A((\hat{A})_{\leq n}) + \mu_B((T)_{\leq n})} \right) \cdot (1 - \varepsilon)^{k-1} \text{ for odd } k$$

and

$$\rho_{\mu}^{n,k}(\Sigma_{\text{uns}}, \Sigma) < \left(\frac{\mu_A((\hat{A})_{\leq n}) + \mu_A((S_{\text{uns}})_{\leq n})}{\mu_A((\hat{A})_{\leq n}) + \mu_A((S)_{\leq n})} \right) \cdot (1 - \varepsilon)^{k-1} \text{ for even } k.$$

In both cases the limit of frequencies $\rho_{\mu}^{n,k}(\Sigma_{\text{uns}}, \Sigma)$ while $d(n, k) \rightarrow \infty$ exists, equal to zero and doesn't depend on a particular choice of this direction. Moreover, for large enough n and k these frequencies bounded by $(1 - \varepsilon)^{k-1}$ and this completes the prove.

The proof of the theorem for \mathcal{CRF} is straightforward. •

4.4 Regular and stable normal forms: L -measure and λ_L -measure

Now we are ready to formulate and prove one of the most important results of this work.

Theorem B. *Let $G = A *_C B$ be an amalgamated product, where A, B, C are free groups of finite rank. If C of infinite index either in A or in B , then sets of all unstable $\mathcal{NF}_{\text{uns}}$ and all singular $\mathcal{NF}_{\text{sin}}$ normal forms are exponentially $\lambda_{\mathcal{NF}}$ -measurable, where $\mathcal{NF} = \{\mathcal{EF}, \mathcal{RF}, \mathcal{CNF}, \mathcal{CRF}\}$.*

Proof. Observe, that as in Theorem A it is sufficient to show the result only for unstable normal forms. Suppose C has an infinite index in A .

Consider the set of all freely-reduced normal forms first. Since C has an infinite index in A , there exists at least one stable representative $s \in S_{\text{st}}$. Set $W = \{y_i s y_j | y_i, y_j \in Y \cup Y^{-1}\}$; and let, as in Lemma 4.4, notation F_0 mean the set of all words in $F = A * B$, that doesn't contain any element of W as a subword, and let F_1 be the supplement of F_0 in F .

Due to Lemma 4.5, the set of all unstable freely-reduced forms can be written as $\mathcal{EF}_{\text{uns}} = (\bigcup_{s \in S_{\text{uns}}} Cs) * (\bigcup_{t \in T_{\text{uns}}} Ct)$ and therefore doesn't contain words having subwords of the type $y_i s y_j$, i.e. $\mathcal{EF}_{\text{uns}} \subseteq F_0$. The set $\mathcal{EF} = A * B$ is a free group and all cones $C(u)$ in $A * B$ are precisely all reduced words in this group that start from u . Obviously, all such cones $C(u)$ have nontrivial intersection with F_1 , and by Lemmata 4.5 and 4.4 it follows that unstable freely-reduced forms are exponentially $\lambda_{\mathcal{EF}}$ -measurable.

Let us consider the set of all nontrivial reduced forms and the subset of all unstable forms in it. By Lemma 4.5 we have: $\mathcal{RF}_{\text{uns}} = (A'_1 \setminus C) * (B'_1 \setminus C)$, where $A'_1 = \bigcup_{s \in S_{\text{uns}}} Cs$ and $B'_1 = \bigcup_{t \in T_{\text{uns}}} Ct$. Therefore, the set of all unstable forms is regular and $\mathcal{RF}_{\text{uns}} \subseteq F_0$.

The set of all reduced forms $\mathcal{RF} = (A \setminus C) * (B \setminus C)$ is regular and prefix-closed in $A * B$ by lemma 4.5, and since $C \cap F_1 = \emptyset$, the set \mathcal{RF} has nonempty intersection with F_1 . Then by Lemma 4.4 the set $\mathcal{RF}_{\text{uns}}$ is exponentially $\lambda_{\mathcal{RF}}$ -measurable.

Let us prove the theorem for \mathcal{CNF} given by formulae (20) and (21):

$$\mathcal{CNF} = (A \circ (S * T)_1) \sqcup (B \circ (S * T)_2) \text{ and } \mathcal{CNF}_{\text{uns}} = (A \circ (S_{\text{uns}} * T_{\text{uns}})_1) \sqcup (B \circ (S_{\text{uns}} * T_{\text{uns}})_2).$$

As in Theorem A above, we prove the statement of this theorem for the pair $\Sigma = (A \circ (S * T)_1)$ and $\Sigma_{\text{uns}} = (A \circ (S_{\text{uns}} * T_{\text{uns}})_1)$.

The set Σ_{uns} is a (regular) subset of F_0 . To show that Σ_{uns} doesn't contain non-small Σ -cones, decompose both sets using (14). Then

$\Sigma_{\text{uns}} = A(T_{\text{uns}} \cup T_{\text{uns}} S_{\text{uns}} T_{\text{uns}} \cup T_{\text{uns}} (S_{\text{uns}} T_{\text{uns}})^* S_{\text{uns}})$ and $\Sigma = A(T \cup T S T \cup T (S T)^* S)$.

All cones in Σ , except, may be, Σ -cones from AT , have nonempty intersection with F_1 and so they can't be contained in Σ_{uns} ; but AT doesn't have non-small Σ -cones, and therefore, the set $\mathcal{CNF}_{\text{uns}}$ is exponentially $\lambda_{\mathcal{CNF}}$ -measurable.

The proof of the theorem for \mathcal{CRF} is straightforward. •

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